FAKULTÄT FÜR INFORMATIK
Faculty of Informatics

Ranking Specific Sets of Objects

Jan Maly, Stefan Woltran

PPI17 @ BTW 2017, Stuttgart

March 7, 2017

Lifting rankings from objects to sets

Given

- A set S
- A linear order $<$ on S
- A family $\mathcal{X} \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$ of nonempty subsets of S

The problem

Is there a “good” ranking $\prec$ on $\mathcal{X}$?

An example - Basics

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\},$ $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$

The axiomatic approach

What is a "good" ranking?

- The ranking should be based on the linear order $<$
- The ranking should be transitive, either reflexive or irreflexive, ...
- "good" depends on the interpretation of $\mathcal{X}$

Possible interpretations

- Sets as final outcomes
- Opportunities
- Complete uncertainty
- etc...

Axioms for ranking sets under complete uncertainty

Extension Rule

For all $x, y \in S$ if $\{x\}, \{y\} \in \mathcal{X}$, then

$$\{x\} \prec \{y\} \text{ iff } x < y$$

Dominance

For all $A \in \mathcal{X}$ and all $x \in S$ if $A \cup \{x\} \in \mathcal{X}$, then

$$y < x \text{ for all } y \in A \text{ implies } A \prec A \cup \{x\}$$

$$x < y \text{ for all } y \in A \text{ implies } A \cup \{x\} \prec A$$

If $\mathcal{X} = \mathcal{P}(S) \setminus \{\emptyset\}$ and $\prec$ is transitive, the extension rule is implied by dominance.

An example - extension rule and dominance

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\},$ $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$

An example - extension rule and dominance

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\},$ $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$
- Extension rule: $\{$strawberry$\} \prec \{$chocolate$\}$

An example - extension rule and dominance

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\}$, $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$
- Extension rule: $\{$strawberry$\} \prec \{$chocolate$\}$
- Dominance:
  $\{$strawberry$\} \prec \{$strawberry, vanilla$\} \prec \{$strawberry, vanilla, lemon$\}$
  $\{$strawberry$\} \prec \{$strawberry, lemon$\}$

Axioms for ranking sets under complete uncertainty

Independence

For all $A, B \in \mathcal{X}$ and for all $x \in S \setminus (A \cup B)$ if $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$, then

$$A \prec B \text{ implies } A \cup \{x\} \preceq B \cup \{x\}$$

Strict Independence

For all $A, B \in \mathcal{X}$ and for all $x \in S \setminus (A \cup B)$ if $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$, then

$$A \prec B \text{ implies } A \cup \{x\} \prec B \cup \{x\}$$

An example - all axioms

- $S = \{\text{strawberry, vanilla, chocolate, lemon}\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{\text{strawberry}\}, \{\text{chocolate}\}, \{\text{strawberry, vanilla}\}, \{\text{strawberry, vanilla, lemon}\}, \{\text{strawberry, lemon}\}\}$
- Extension rule: $\{\text{strawberry}\} \prec \{\text{chocolate}\}$
- Dominance:
  $\{\text{strawberry}\} \prec \{\text{strawberry, vanilla}\} \prec \{\text{strawberry, vanilla, lemon}\}$
  $\{\text{strawberry}\} \prec \{\text{strawberry, lemon}\}$

An example - all axioms

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\}$, $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$
- Extension rule: $\{$strawberry$\} \prec \{$chocolate$\}$
- Dominance:
  $\{$strawberry$\} \prec \{$strawberry, vanilla$\} \prec \{$strawberry, vanilla, lemon$\}$
  $\{$strawberry$\} \prec \{$strawberry, lemon$\}$
- Independence: $\{$strawberry, lemon$\} \preceq \{$strawberry, vanilla, lemon$\}$

An example - all axioms

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\}$, $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$
- Extension rule: $\{$strawberry$\} \prec \{$chocolate$\}$
- Dominance:
  $\{$strawberry$\} \prec \{$strawberry, vanilla$\} \prec \{$strawberry, vanilla, lemon$\}$
  $\{$strawberry$\} \prec \{$strawberry, lemon$\}$
- Independence: $\{$strawberry, lemon$\} \preceq \{$strawberry, vanilla, lemon$\}$
- Strict independence:
  $\{$strawberry, lemon$\} \prec \{$strawberry, vanilla, lemon$\}$

Classic impossibility results

Kannai and Peleg (1984)

Assume $\mathcal{X} = \mathcal{P}(S) \setminus \{\emptyset\}$ and $|S| \geq 6$, then there exists no order on $\mathcal{X}$ satisfying dominance and independence.

Barberà and Pattanaik (1984)

Assume $\mathcal{X} = \mathcal{P}(S) \setminus \{\emptyset\}$ and $|S| \geq 3$, then there exists no binary relation on $\mathcal{X}$ satisfying dominance and strict independence.

Proof of Barberà and Pattanaik

- Assume $S = \{1, 2, 3\}$
- (1) $\{1\} \prec \{1, 2\}$ and (2) $\{2, 3\} \prec \{3\}$ by dominance
- $\{1, 3\} \prec \{1, 2, 3\}$ by (1) and strict independence
- $\{1, 2, 3\} \prec \{1, 3\}$ by (2) and strict independence

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{3\}?\{2, 5\}$

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{3\} \prec \{2, 5\}$

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{3\} \prec \{2, 5\}$
- $\{3, 6\} \preceq \{2, 5, 6\}$ by independence

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{3\} \prec \{2, 5\}$
- $\{3, 6\} \preceq \{2, 5, 6\}$ by independence
- $\{3, 4, 5, 6\} \preceq \{2, 3, 4, 5, 6\}$ by observation

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{3\} \prec \{2, 5\}$
- $\{3, 6\} \preceq \{2, 5, 6\}$ by independence
- $\{3, 4, 5, 6\} \preceq \{2, 3, 4, 5, 6\}$ by observation
- This contradicts dominance!

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{2, 5\} \preceq \{3\}$

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{2, 5\} \preceq \{3\} \prec \{4\}$

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{2, 5\} \prec \{4\}$

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{2, 5\} \prec \{4\}$
- $\{1, 4\} \succeq \{1, 2, 5\}$ by independence

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{2, 5\} \prec \{4\}$
- $\{1, 4\} \succeq \{1, 2, 5\}$ by independence
- $\{1, 2, 3, 4\} \succeq \{1, 2, 3, 4, 5\}$ by observation

Proof of Kannai and Peleg

- Observation: dominance and independence imply $A \sim \{\max(A), \min(A)\}$
- Assume $S = \{1, 2, \ldots, 6\}$
- $\{2, 5\} \prec \{4\}$
- $\{1, 4\} \succeq \{1, 2, 5\}$ by independence
- $\{1, 2, 3, 4\} \succeq \{1, 2, 3, 4, 5\}$ by observation
- This contradicts dominance!

Ditching the assumption $\mathcal{X} = \mathcal{P}(S) \setminus \{\emptyset\}$

- In many applications $\mathcal{X}$ is subject to constraints.
- There are families $\mathcal{X} \neq \mathcal{P}(S) \setminus \{\emptyset\}$ with $|S| > 6$ such that there is an order on $\mathcal{X}$ satisfying dominance and (strict) independence.
- It can be argued that dominance is too weak in the general case.

An example - all axioms

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\}$, $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$
- Extension rule: $\{$strawberry$\} \prec \{$chocolate$\}$
- Dominance:
  $\{$strawberry$\} \prec \{$strawberry, vanilla$\} \prec \{$strawberry, vanilla, lemon$\}$
  $\{$strawberry$\} \prec \{$strawberry, lemon$\}$
- Independence: $\{$strawberry, lemon$\} \preceq \{$strawberry, vanilla, lemon$\}$
- Strict independence:
  $\{$strawberry, lemon$\} \prec \{$strawberry, vanilla, lemon$\}$

An example - all axioms

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\}$, $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$
- Extension rule: $\{$strawberry$\} \prec \{$chocolate$\}$
- Dominance:
  $\{$strawberry$\} \prec \{$strawberry, vanilla$\} \prec \{$strawberry, vanilla, lemon$\}$
  $\{$strawberry$\} \prec \{$strawberry, lemon$\}$
- Independence: $\{$strawberry, lemon$\} \preceq \{$strawberry, vanilla, lemon$\}$
- Strict independence:
  $\{$strawberry, lemon$\} \prec \{$strawberry, vanilla, lemon$\}$
- $\{$strawberry, vanilla$\} \overset{?}{\prec} \{$strawberry, lemon$\}$

A strengthening of dominance

- Solution: Define a stronger version of dominance

Maximal dominance

For all $A, B \in \mathcal{X}$,

$$(\max(A) \leq \max(B) \wedge \min(A) < \min(B)) \vee$$
$$(\max(A) < \max(B) \wedge \min(A) \leq \min(B)) \rightarrow A \prec B$$

Assuming $\mathcal{X} = \mathcal{P}(S) \setminus \{\emptyset\}$, maximal dominance is implied by dominance and (strict) independence.

An example - maximal dominance

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\}$, $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$

An example - maximal dominance

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\}$, $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$
- Maximal dominance:
  $\{$strawberry$\} \prec \{$strawberry, vanilla$\} \prec \{$strawberry, vanilla, lemon$\}$
  $\{$strawberry$\} \prec \{$strawberry, lemon$\}$

An example - maximal dominance

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\}$, $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$
- Maximal dominance:
  $\{$strawberry$\} \prec \{$strawberry, vanilla$\} \prec \{$strawberry, vanilla, lemon$\}$
  $\{$strawberry$\} \prec \{$strawberry, lemon$\}$
  $\{$strawberry$\} \prec \{$chocolate$\}$

An example - maximal dominance

- $S = \{$strawberry, vanilla, chocolate, lemon$\}$
- Linear order: strawberry < chocolate < vanilla < lemon
- $\mathcal{X} = \{\{$strawberry$\}, \{$chocolate$\}, \{$strawberry, vanilla$\}$, $\{$strawberry, vanilla, lemon$\}, \{$strawberry, lemon$\}\}$
- Maximal dominance:
  $\{$strawberry$\} \prec \{$strawberry, vanilla$\} \prec \{$strawberry, vanilla, lemon$\}$
  $\{$strawberry$\} \prec \{$strawberry, lemon$\}$
  $\{$strawberry$\} \prec \{$chocolate$\}$
  $\{$strawberry, vanilla$\} \prec \{$strawberry, lemon$\}$

The problems we treated

The Partial (Max)-Dominance-(Strict)-Independence Problem

Given a linearly ordered set S and a set $\mathcal{X} \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$, decide if there is a partial order/preorder on $\mathcal{X}$ satisfying (maximal) dominance and (strict) independence.

The (Max)-Dominance-(Strict)-Independence Problem

Given a linearly ordered set S and a set $\mathcal{X} \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$, decide if there is a (strict) total order on $\mathcal{X}$ satisfying (maximal) dominance and (strict) independence.

The Partial (Max)-Dominance-(Strict)-Independence Problem

Theorem

The Partial (Max)-Dominance-Independence Problem is trivial.

We can define a preorder satisfying maximal dominance and independence.

The Partial (Max)-Dominance-(Strict)-Independence Problem

Theorem

The Partial (Max)-Dominance-Independence Problem is trivial.

We can define a preorder satisfying maximal dominance and independence.

Theorem

The Partial (Max)-Dominance-Strict-Independence Problem is P-complete.

- We construct the minimal transitive relation satisfying (maximal) dominance and strict independence, then check if this relation is irreflexive.
- We can prove the P-hardness by a reduction from Horn-Sat.

The Total (Max)-Dominance-(Strict)-Independence Problem

- The (Max)-Dominance-(Strict)-Independence problem is NP-hard.
- This can be shown via a reduction from betweenness.

The Betweenness Problem

Given a finite set $V = \{v_1, v_2, \ldots, v_n\}$ and a set of triples $R \subseteq V^3$, find a strict total order on V such that $a < b < c$ or $a > b > c$ holds for all $(a, b, c) \in R$.

The NP-hardness of betweenness was shown 1979 by Jaroslav Opatrny.

The Total Max-Dominance-Strict-Independence Problem

- Idea: Represent the elements $v_1, v_2, \ldots, v_n$ of V by sets $V_1, V_2, \ldots, V_n$.
- $V_i := \{1, N\} \cup \{i+1, i+2, \ldots, N-i\}$ for sufficiently large N.
- All sets have the same maximal and minimal element.
- The second largest elements are decreasing and second smallest elements are increasing.

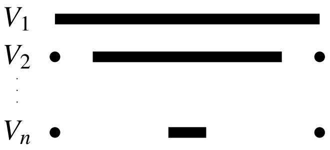

Figure: Sketch of the sets V_1, V_2 and V_n

The Total Max-Dominance-Strict-Independence Problem

- For the triple $(a, b, c) \in R$ represented by the sets A, B, C we add the following sets, where k is unique for this triple: $A \setminus \{k\}$, $B \setminus \{k\}$, $B \setminus \{k+1\}$, $C \setminus \{k+1\}$, $A \setminus \{k+2\}$, $B \setminus \{k+2\}$, $B \setminus \{k+3\}$, $C \setminus \{k+3\}$
- We want $B \setminus \{k+1\} \prec A \setminus \{k\}$, $B \setminus \{k\} \prec C \setminus \{k+1\}$, $A \setminus \{k+2\} \prec B \setminus \{k+3\}$ and $C \setminus \{k+3\} \prec B \setminus \{k+2\}$

The Total Max-Dominance-Strict-Independence Problem

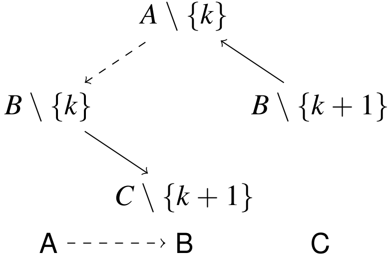

Figure: Family that forces that $A \prec B$ leads to $B \prec C$

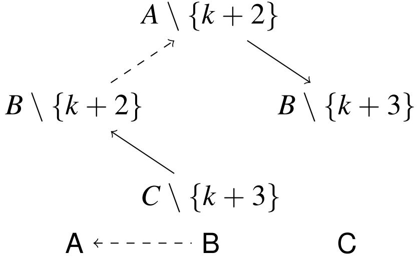

Figure: Family that forces that $A \succ B$ leads to $B \succ C$

The Total Max-Dominance-Strict-Independence Problem

- For the triple $(a, b, c) \in R$ represented by the sets A, B, C we add the following sets, where k is unique for this triple: $A \setminus \{k\}$, $B \setminus \{k\}$, $B \setminus \{k+1\}$, $C \setminus \{k+1\}$, $A \setminus \{k+2\}$, $B \setminus \{k+2\}$, $B \setminus \{k+3\}$, $C \setminus \{k+3\}$
- We want $B \setminus \{k+1\} \prec A \setminus \{k\}$, $B \setminus \{k\} \prec C \setminus \{k+1\}$, $A \setminus \{k+2\} \prec B \setminus \{k+3\}$ and $C \setminus \{k+3\} \prec B \setminus \{k+2\}$
- For example, we can force $B \setminus \{k+1\} \prec A \setminus \{k\}$ by adding $A \setminus \{k, k+4\}$, $B \setminus \{k+1, k+4\}$ and either $A \setminus \{1, k, k+4\}$, $B \setminus \{1, k+1, k+4\}$ or $A \setminus \{k, k+4, N\}$, $B \setminus \{k+1, k+4, N\}$

A summary of our results

	Not total	Total
Dom + Ind	always	NP-complete
Max Dom +Ind	always	NP-complete
Dom + Strict Ind	in P	NP-complete
Max Dom + Strict Ind	in P	NP-complete

Future work

- The complexity of the studied problems if $\mathcal{X}$ is given in a compact way.
- Characterize the sets $\mathcal{X}$ that have orders satisfying (maximal) dominance and (strict) independence.
- Study other axioms and interpretations.